LAWRENCE LIVERMORE NATIONAL LABORATORY

LLNL-CONF-674194

Evaluation of Sommerfeld Integrals Using Adaptive Filon-Type Integration

G. J. Burke

July 7, 2015

2015 IEEE International Symposium on Antennas and Propagation and North American Radio Science Meeting
Vancouver, BC, Canada
July 19, 2015 through July 25, 2015

Evaluation of Sommerfeld Integrals Using Adaptive Filon-Type Integration

G. J. Burke
Lawrence Livermore National Lab.

2015 IEEE International Symposium on
Antennas and Propagation and
North American Radio Science Meeting

Vancouver, BC, Canada
19-25 July, 2015

LLNL-PRES-?????

This work was performed under the auspices of the U.S. Department of Energy by Lawrence Livermore National Laboratory under Contract DE-AC52-07NA27344

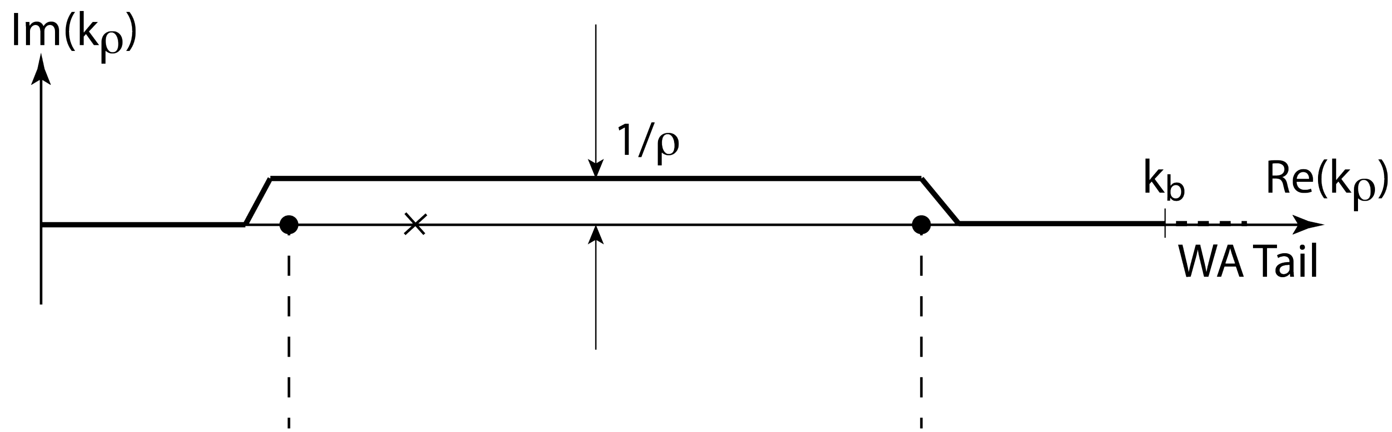
Green's functions for ground involve highly oscillatory integrands

Sommerfeld integrals are usually integrated on the real axis with deformation around singularities and terminated with a convergence acceleration method such as weighted averages.

When ρ is large integration from 0 to k_b is difficult due to many oscillations of the Bessel function and the contour forced close to the singularities.

$$I_\nu(\rho, z, z') = \int_0^\infty G(z, z'; k_\rho) J_\nu(\rho k_\rho) k_\rho \, dk_\rho$$

Contour:



Filon's method incorporates oscillations into the integration rule

Function $f(k_\rho)$ is approximated with piece-wise quadratics and the product with the oscillating function integrated analytically

$$S_\nu \rho) = \int_{k_1}^{k_3} f(k_\rho) J_\nu(\rho k_\rho) k_\rho \, dk_\rho \approx \int_{k_1}^{k_3} f_s(k_\rho) J_\nu(\rho k_\rho) k_\rho \, dk_\rho$$

$$f_s(k_\rho) = C_1 + C_2(k_\rho - k_2) + C_3(k_\rho - k_2)^2$$

$$C_1 = f(k_2) = f_2$$

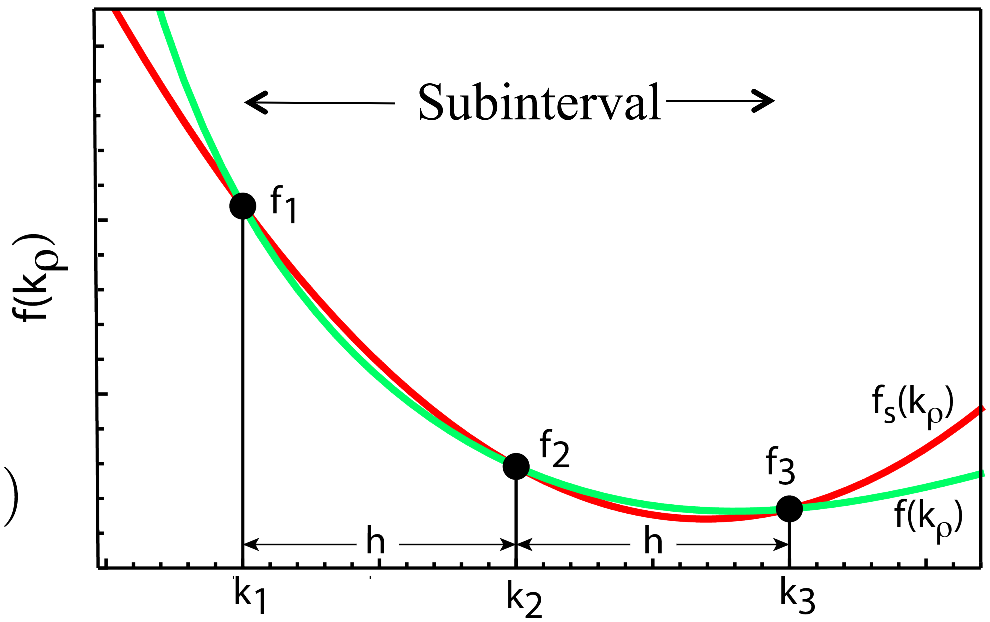
$$C_2 = \frac{1}{2h}(f_3 - f_1)$$

$$C_3 = \frac{1}{2h^2}(f_1 - 2f_2 + f_3)$$

Will need f_s' and f_s''

$$f_s'(k_\rho) = C_2 + 2C_3(k_\rho - k_2)$$

$$f_s'' = 2C_3$$



L. Filon, Proc. Roy. Soc. Edinburgh 49, 38-47, 1928

Filon Hankel transform: Barakat and Parshall, Appl. Math. Lett., 1996

For Filon's method the integral is integrated by parts twice

$$\tilde{S}_0(\rho) = \int_a^b f_s(k_\rho) \underbrace{J_0(\rho k_\rho) k_\rho}_{\mathrm{P}_0} \, dk_\rho$$

For Filon, substitute the quadratic approximation and integrate by parts twice

$$\tilde{S}_0(\rho) = \frac{f_s(k_\rho)}{\rho^2} P_1(\rho k_\rho)\Big|_a^b - \frac{f_s'(k_\rho)}{\rho^3} P_2(\rho k_\rho)\Big|_a^b + \frac{f_s''}{\rho^4} P_3(\rho k_\rho)\Big|_a^b$$

The first limit terms cancel between subintervals, so are dropped for now

$$\bar{S}_0(\rho) = \left[-\frac{f_s'(k_\rho)}{\rho^3} P_2(\rho k_\rho) + \frac{f_s''}{\rho^4} P_3(\rho k_\rho) \right]_a^b$$

$$P_1(z) = \int_0^z J_0(z) z \, dz = z J_1(z) \qquad P_n(z) = \int_0^z P_{n-1}(z) \, dz$$

The J_1 integrals are handled the same way

$$\tilde{S}_1(\rho) = \int_a^b f_s(k_\rho)\underbrace{J_1(\rho k_\rho)k_\rho}_{P_1}\,dk_\rho$$

For Filon, substitute the quadratic approximation and integrate by parts twice

$$\tilde{S}_1(\rho) = \frac{f_s(k_\rho)}{\rho^2}P_2(\rho k_\rho)\Big|_a^b - \frac{f_s'(k_\rho)}{\rho^3}P_3(\rho k_\rho)\Big|_a^b + \frac{f_s''}{\rho^4}P_4(\rho k_\rho)\Big|_a^b$$

Dropping the first limit term, will evaluate

$$\bar{S}_1(\rho) = \left[-\frac{f_s'(k_\rho)}{\rho^3}P_3(\rho k_\rho) + \frac{f_s''}{\rho^4}P_4(\rho k_\rho)\right]_a^b$$

$$P_1(z) = \int_0^z J_0(z)z\,dz = zJ_1(z) \qquad P_n(z) = \int_0^z P_{n-1}(z)\,dz$$

The integrals can be evaluated as Bessel and Struve functions

$\mathbf{H}_n(z)$ = Struve function (Wolfram Research, Mathematica)

$$P_1(z) = \int_0^z z J_0(z)\, dz = z J_1(z)$$

$$P_2(z) = \int_0^z P_1(z)\, dz = \frac{\pi z}{2}\Big(J_1(z)\mathbf{H}_0(z) - J_0(z)\mathbf{H}_1(z)\Big)$$

$$P_3(z) = \int_0^z P_2(z)\, dz$$

$$= z^2 J_0(z) - 2z J_1(z) + \frac{\pi z^2}{2}\Big(J_1(z)\mathbf{H}_0(z) - J_0(z)\mathbf{H}_1(z)\Big)$$

$$P_4(z) = \int_0^z P_3(z)\, dz$$

$$= \frac{z^3}{2} J_0(z) - \frac{z^2}{2} J_1(z) + \frac{\pi z}{4}(z^2 - 3)\Big(J_1(z)\mathbf{H}_0(z) - J_0(z)\mathbf{H}_1(z)\Big)$$

For numerical evaluation use series and asymptotic approximations

For $|z| < 18$ integrate terms of the J_0 series and sum

$$P_0(z) = zJ_0(z) \approx z \sum_{k=0}^{N} \frac{(-1)^k (z/2)^{2k}}{(k!)^2} \qquad P_2(z) \approx \frac{z^3}{2} \sum_{k=0}^{N} \frac{(-1)^k (z/2)^{2k}}{(3+2k)k!(k+1)!}$$

For $|z| > 18$ use the J, **H** form with the asymptotic approximation for **H** and the J, Y Wronskin to eliminate cancelling terms

$$\mathbf{H}_j(z) \sim Y_j(z) + T_j(z) \qquad J_1(z)Y_0(z) - J_0(z)Y_1(z) = 2/(\pi z)$$

$$\begin{aligned} P_2(z) &= \frac{\pi z}{2}\Big(J_1(z)\mathbf{H}_0(z) - J_0(z)\mathbf{H}_1(z)\Big) \\ &\sim 1 + \frac{\pi z}{2}\Big[J_1(z)T_0(z) - J_0(z)T_1(z)\Big] \end{aligned}$$

$$T_0(z) = \frac{2}{\pi}\left[\frac{1}{z} - \frac{1}{z^3} + \frac{1^2\cdot 3^2}{z^5} - \frac{1^2\cdot 3^2\cdot 5^2}{z^7} + \cdots\right]$$

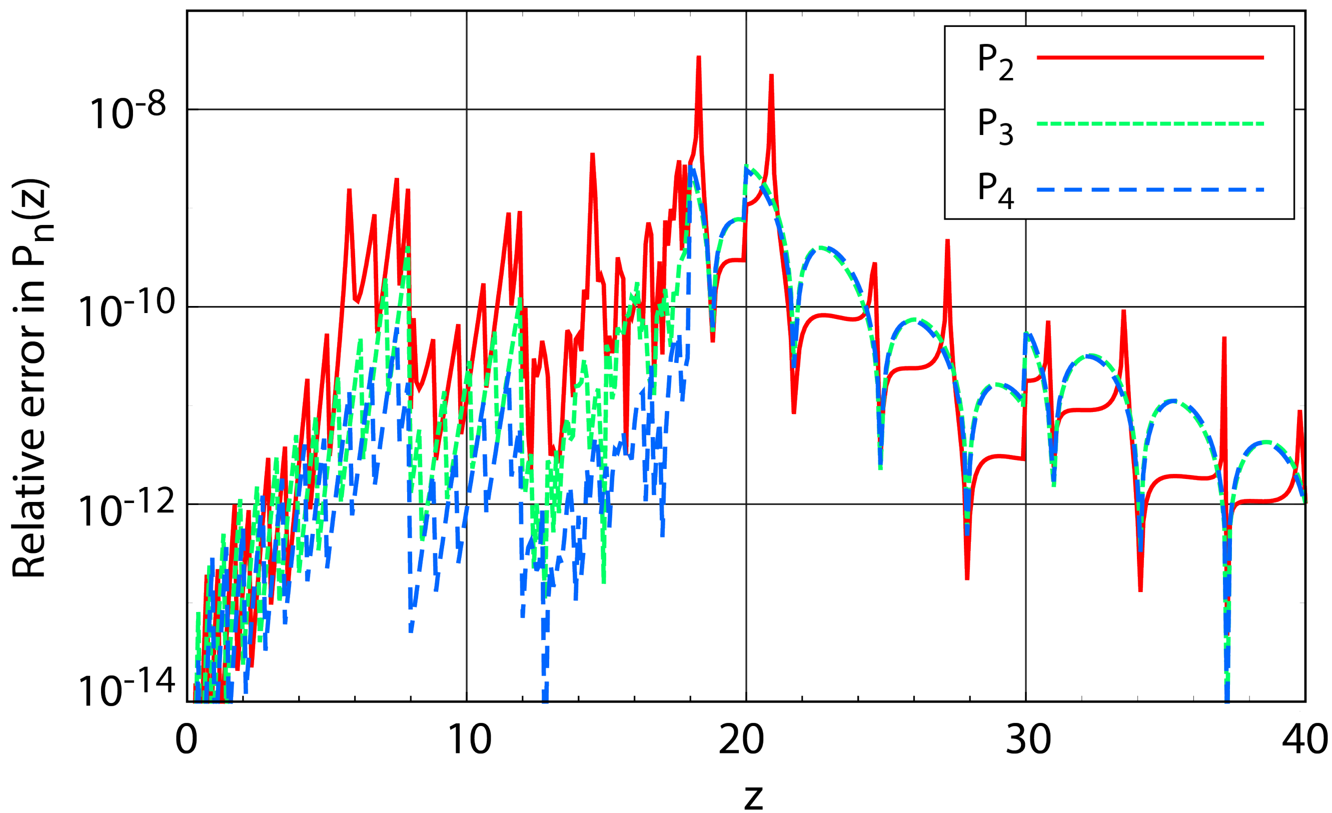
$$T_1(z) = \frac{2}{\pi}\left[1 + \frac{1}{z^2} - \frac{1^2\cdot 3}{z^4} + \frac{1^2\cdot 3^2\cdot 5}{z^6} - \cdots\right]$$

(Abramowitz and Stegun)

Relative errors in P_n integrals are $< 10^{-8}$

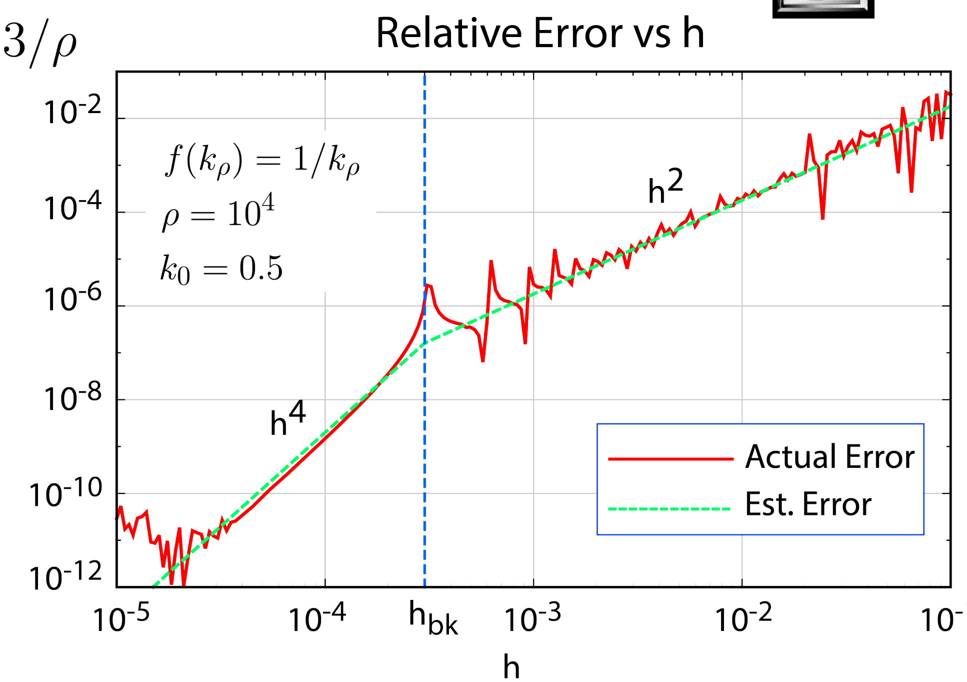
Series for $|z| < 18$

Asymptotic for $|z|>18$



Filon error for the Hankel transform was determined empirically

The behavior changes for $h \gtrless h_{\mathrm{bk}} \approx 3/\rho$



Estimated integral:

$$|I| \sim \frac{k_0^{1/2}}{\rho^{5/2}} |f'(k_0)| \times \begin{cases} 1 & \text{if } h > h_{\mathrm{bk}} \\ h/h_{\mathrm{bk}} & \text{otherwise} \end{cases}$$

Estimated error:

$$E \sim \frac{k_0^{1/2}}{\rho^{5/2}} |f^{(3)}(k_0)| \times \begin{cases} h^2 & \text{if } h > h_{\mathrm{bk}} \\ h^5/h_{\mathrm{bk}}^3 & \text{otherwise} \end{cases}$$

Estimated relative error:

$$E_R \approx 0.3 \frac{|f^{(3)}(k_0)|}{|f'(k_0)|} \times \begin{cases} h^2 & \text{if } h > h_{\mathrm{bk}} \\ h^4/h_{\mathrm{bk}}^2 & \text{otherwise} \end{cases}$$

With a 5-point subinterval (2 Simpson panels)

$$f' \approx \mathrm{Max} \begin{cases} (f_3 - f_1)/(2h) \\ (f_5 - f_3)/(2h) \end{cases} \qquad f^{(3)} \approx \mathrm{Max} \begin{cases} (-f_1 + 3f_2 - 3f_3 + f_4)/h^3 \\ (-f_2 + 3f_3 - 3f_4 + f_5)/h^3 \end{cases}$$

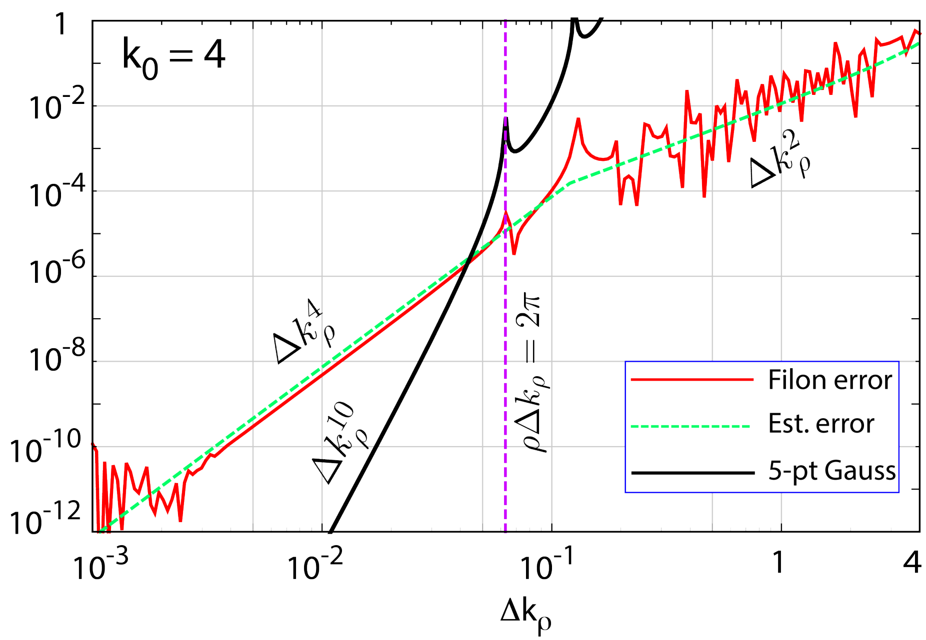
Error was tested for the Sommerfeld Identity integral: $\rho = 10^2$

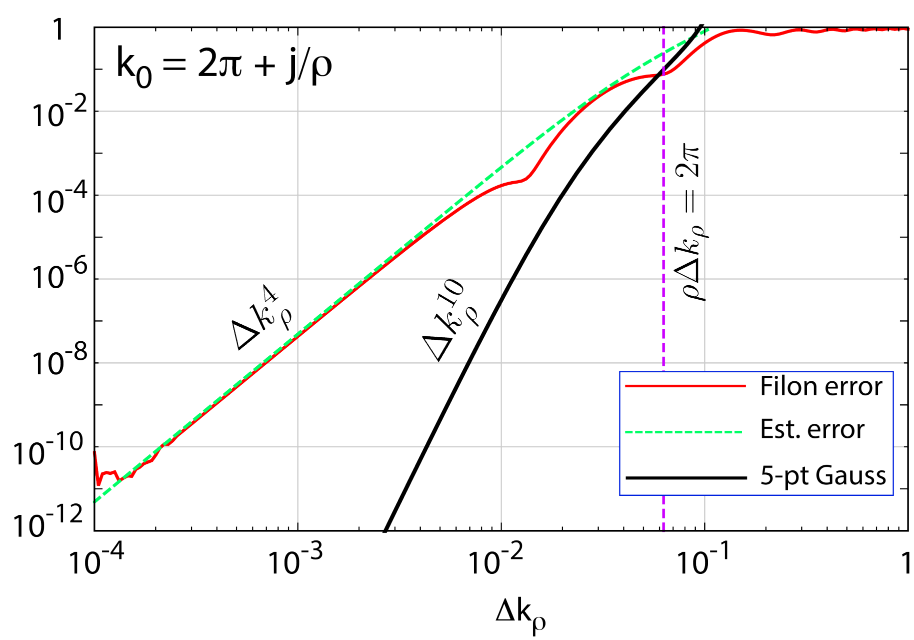
$$\frac{e^{-jkR}}{R} = \int_0^\infty f_{SI}(k_\rho) J_0(\rho k_\rho) k_\rho \, dk_\rho \qquad f_{\mathrm{SI}}(k_\rho) = \frac{e^{-|z|\sqrt{k_\rho^2-(2\pi)^2}}}{\sqrt{k_\rho^2-(2\pi)^2}}$$

$$R = \sqrt{\rho^2 + z^2}$$

Error reference: Mathematica, NIntegrate

Relative error $\qquad \rho = 10^2 \qquad \Delta k_\rho = 4\,h$ = length of 5-point subinterval



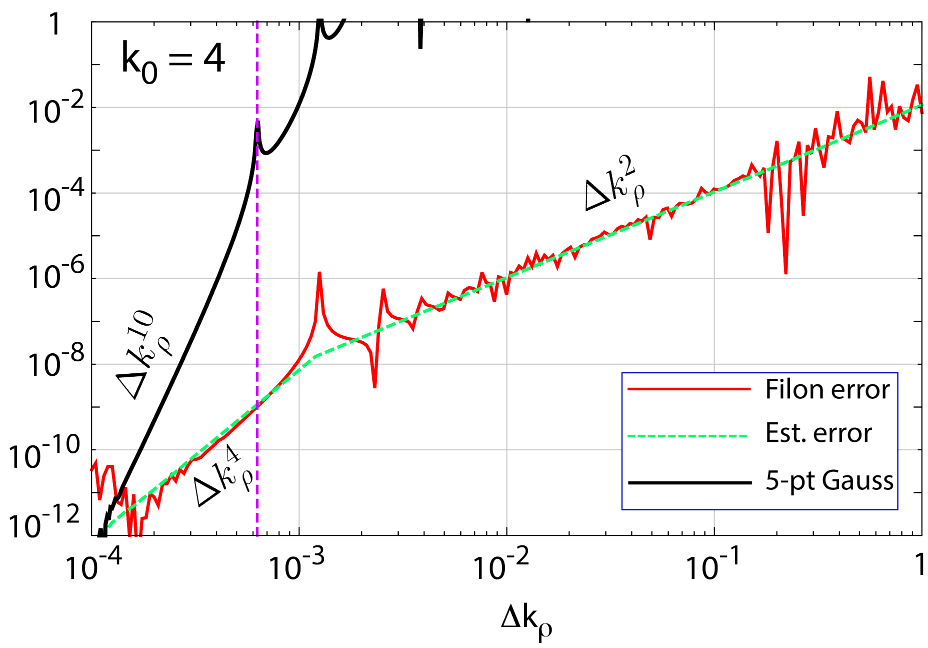


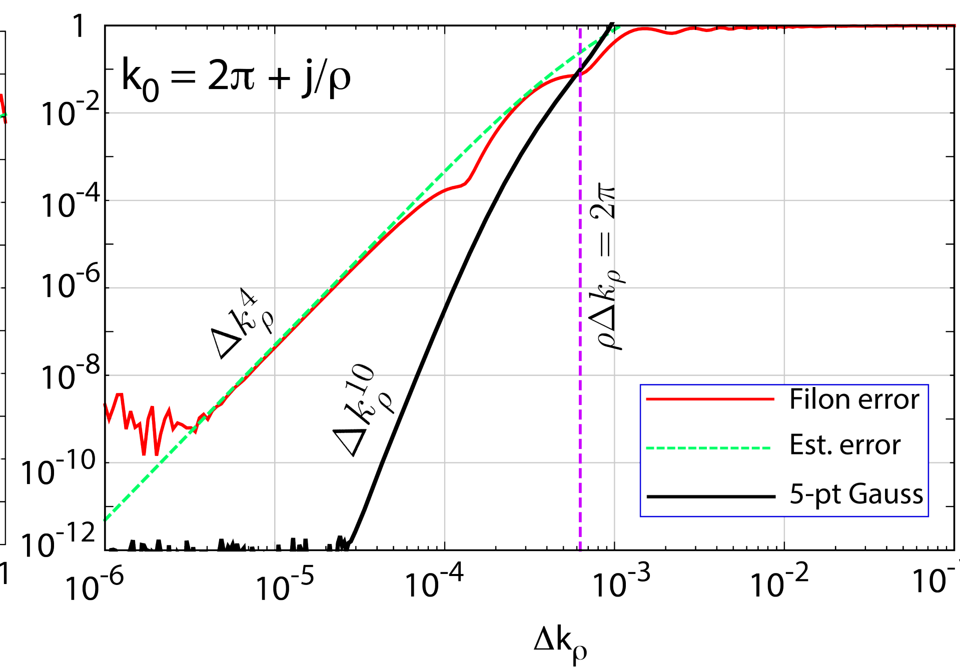
Error was tested for the Sommerfeld Identity integral: $\rho = 10^4$

$$\frac{e^{-jkR}}{R} = \int_0^\infty f_{SI}(k_\rho) J_0(\rho k_\rho) k_\rho \, dk_\rho \qquad f_{\mathrm{SI}}(k_\rho) = \frac{e^{-|z|\sqrt{k_\rho^2-(2\pi)^2}}}{\sqrt{k_\rho^2-(2\pi)^2}}$$

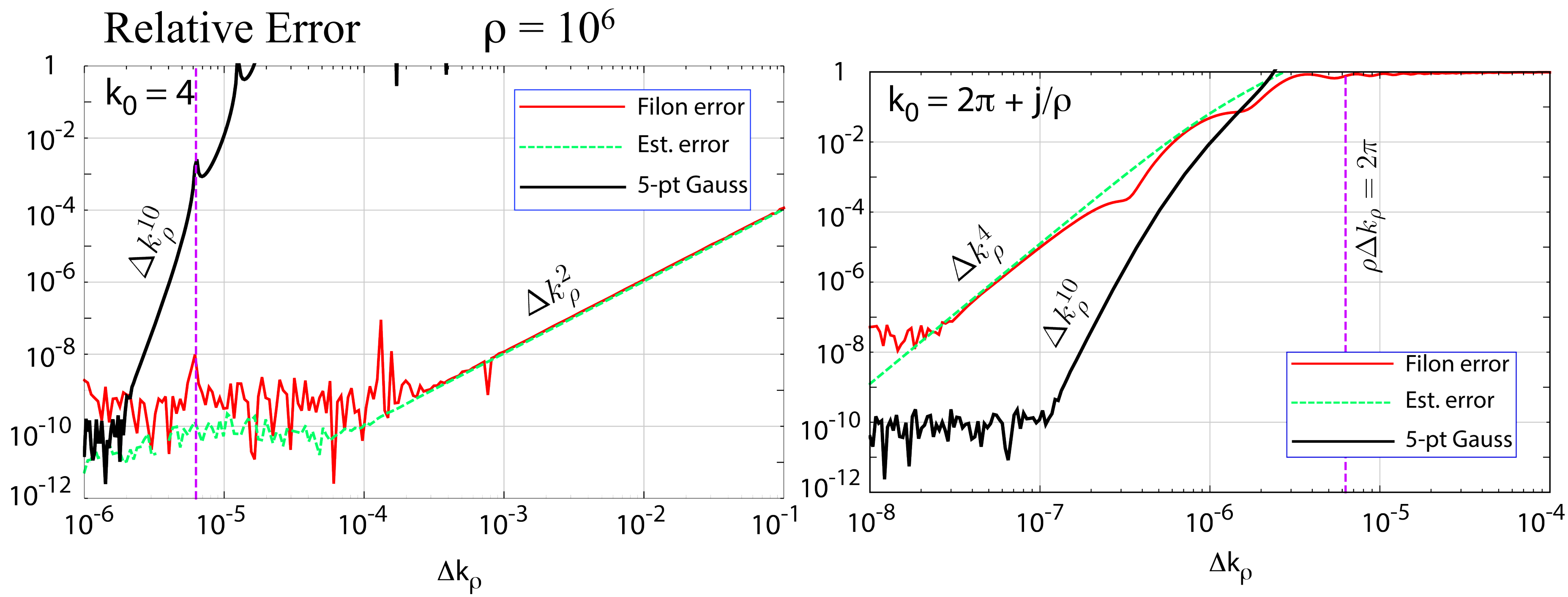
$$R = \sqrt{\rho^2 + z^2}$$

Relative Error $\qquad \rho = 10^4$





Error was tested for the Sommerfeld Identity integral: $\rho = 10^6$



If Filon does not meet the error test by $\rho\Delta k_\rho = \pi$, Patterson's adaptive algorithm is used for the subinterval

(T.N.L Patterson, Communications ACM, pp. 694-699, Nov. 1973)

Error vs k_ρ is demonstrated for the Sommerfeld Identity integral

$\rho = 10^4$

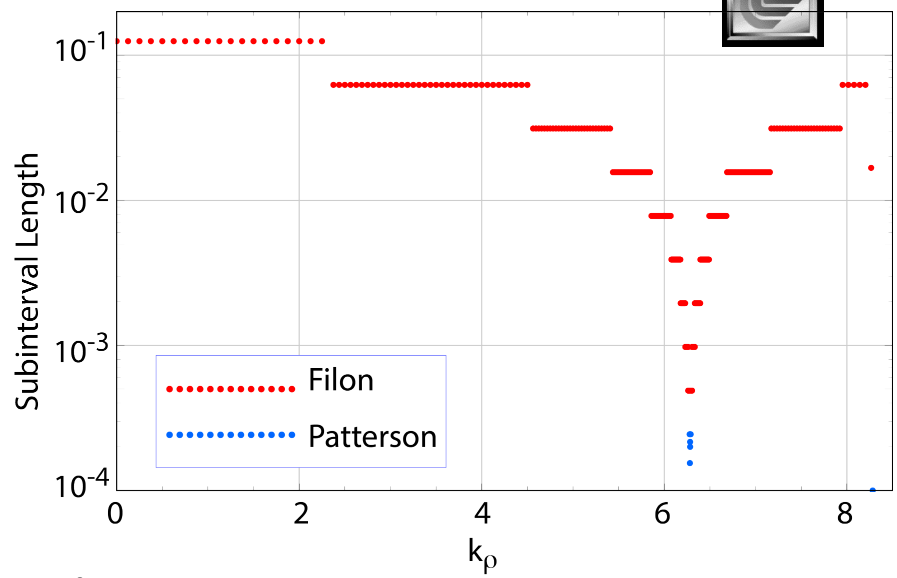
Base: k_ρ = 0 to 8.28 + WA tail

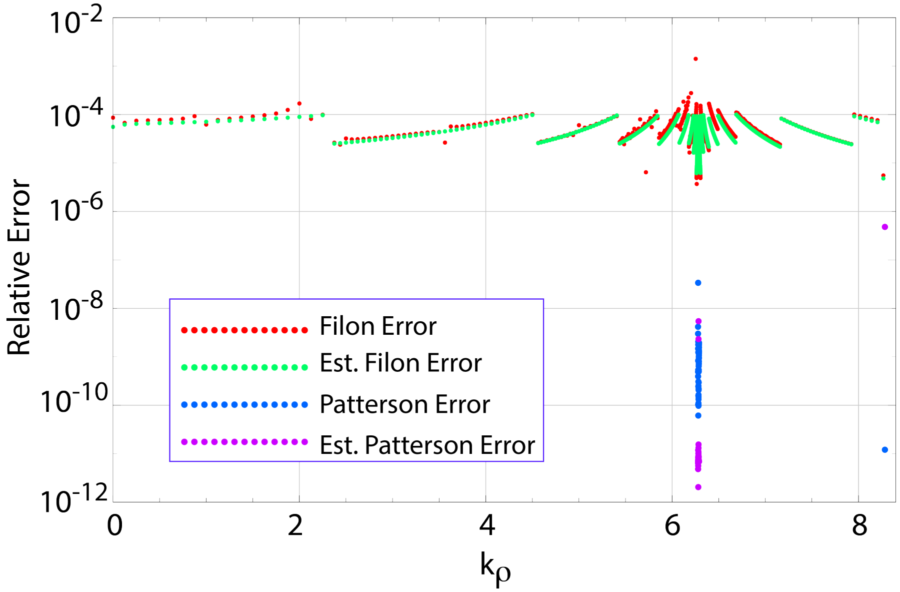
Relative errors:

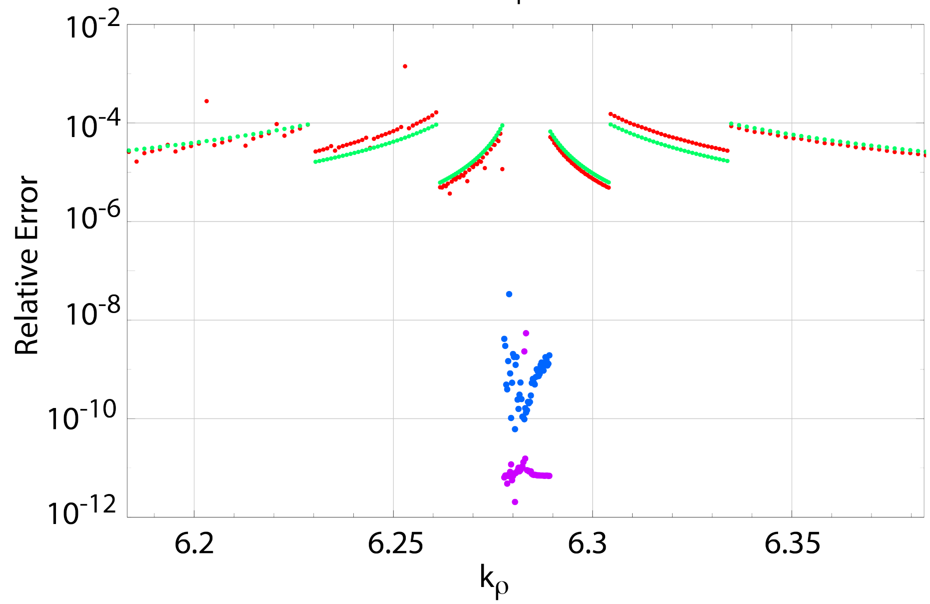
Sommerfeld Identity: $8.3(10^{-8})$

Filon int. only: $8.2(10^{-5})$, goal: 10^{-4}

Patterson only: $1.4(10^{-10})$, goal 10^{-6}





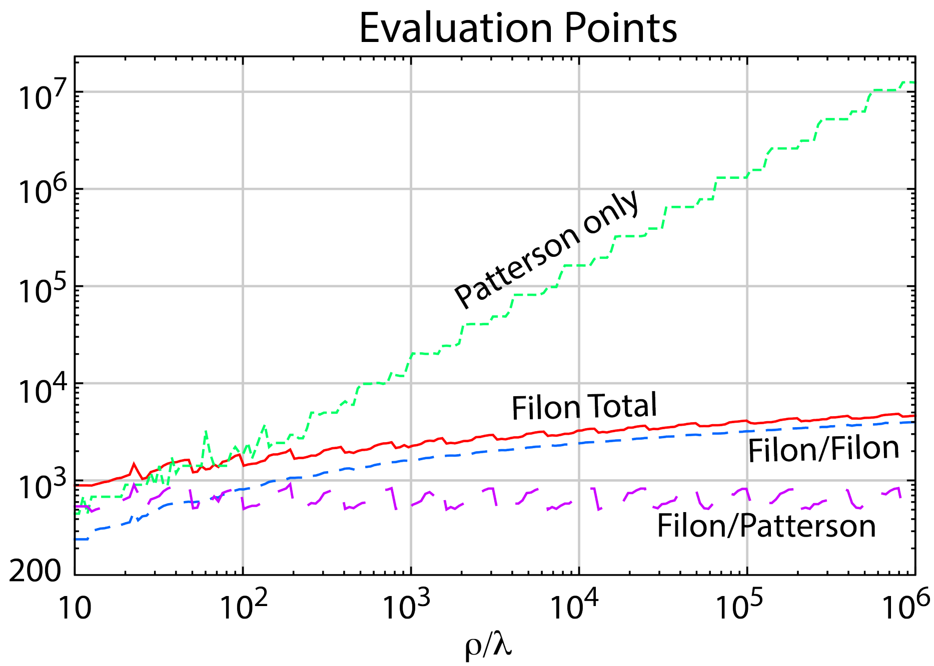


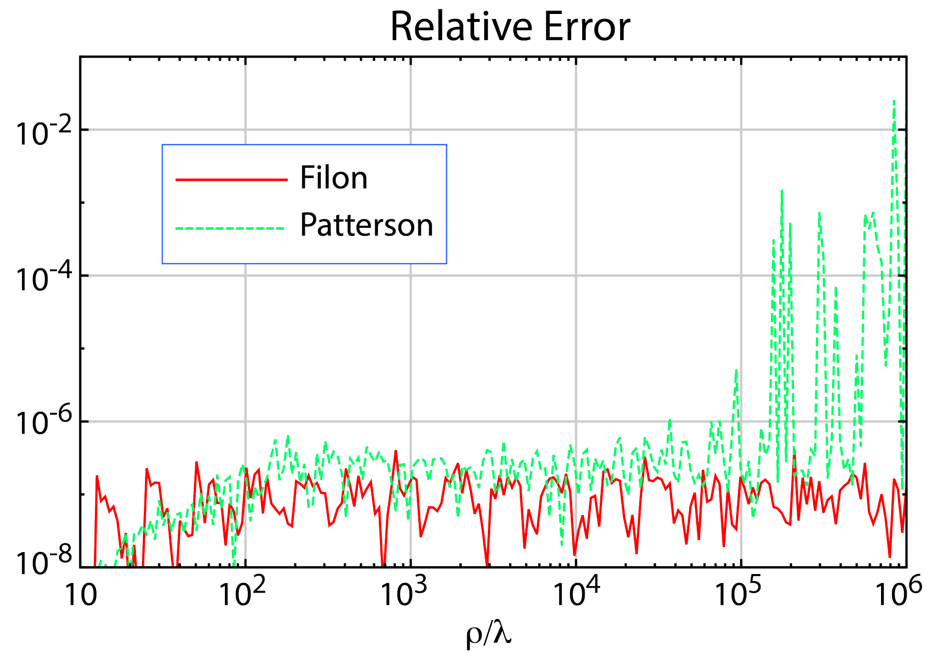
Filon and Patterson integration for the SI are compared vs ρ

For the Sommerfeld Identity:

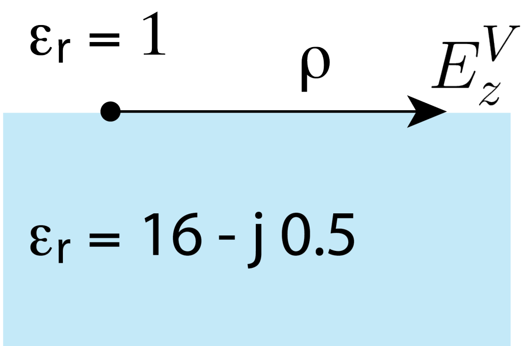
The number of evaluations using Patterson increases linearly with ρ

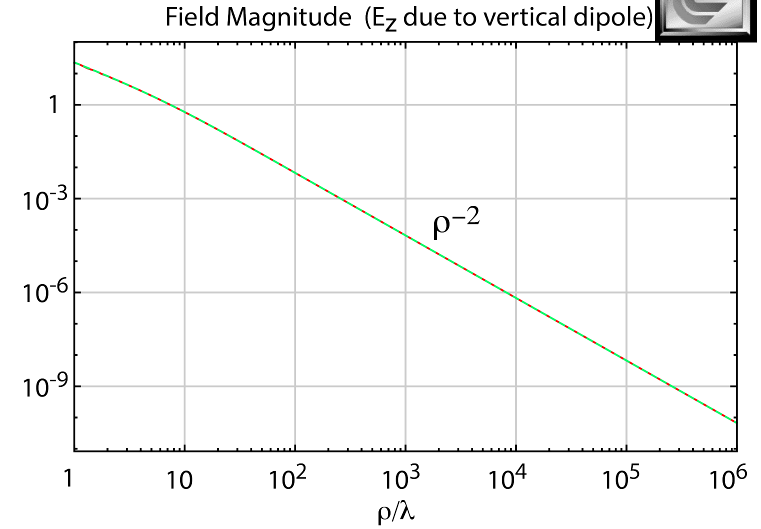
Filon increases more slowly



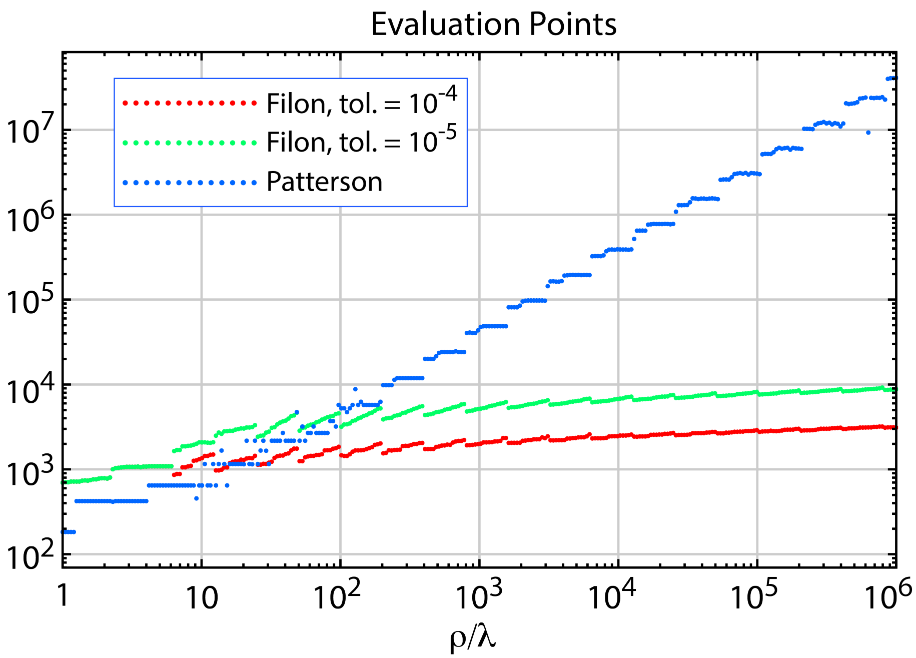


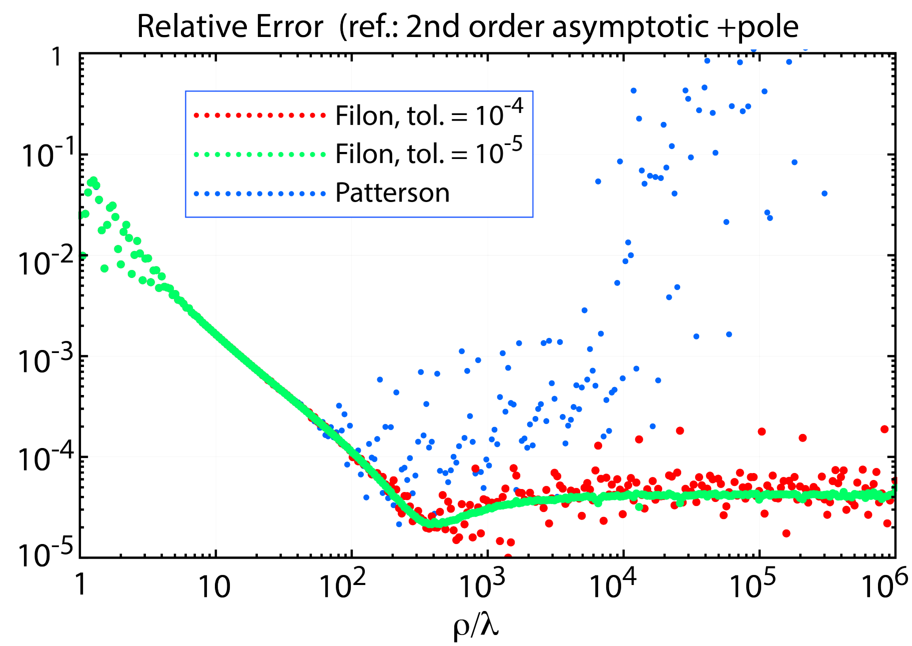
E_z^V on a half-space was compared with an asymptotic approx.



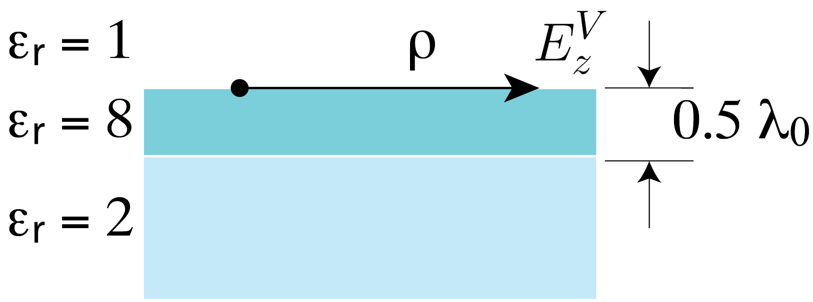


2nd order asymptotic + pole, no 2nd BC



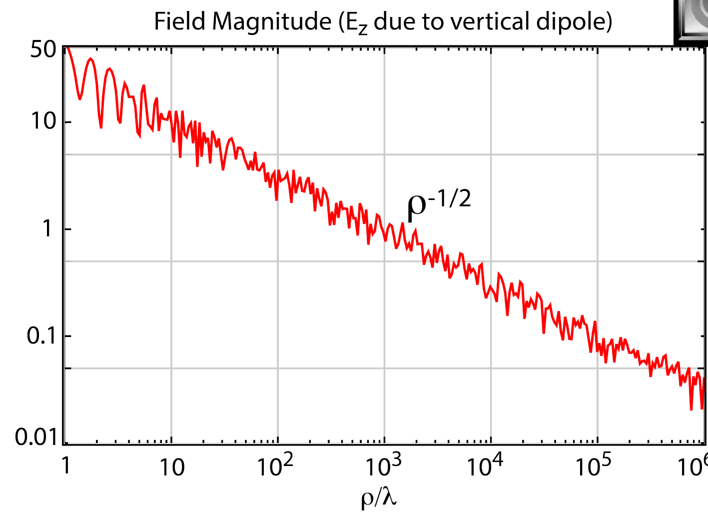


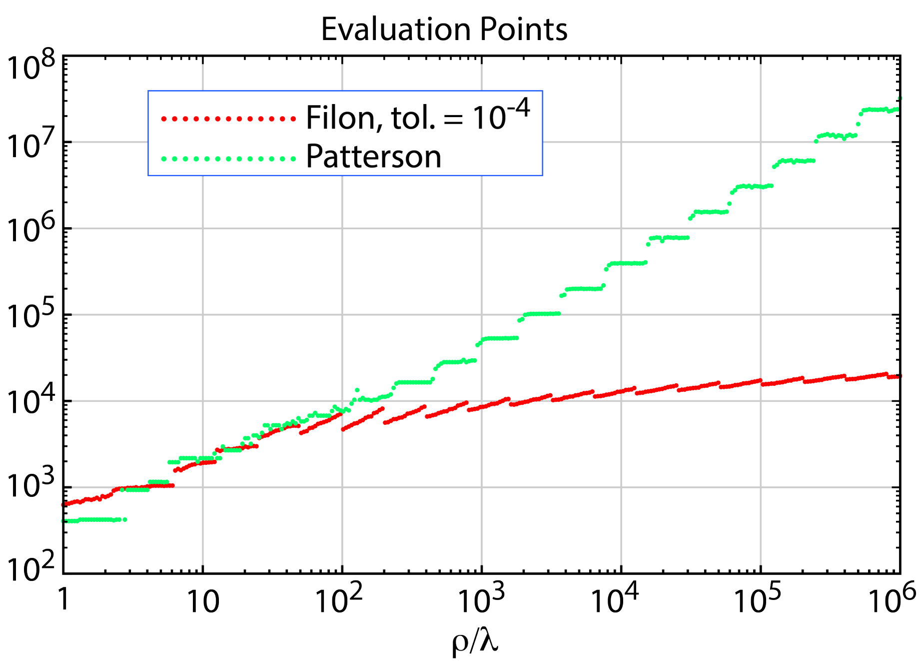
On a dielectric slab the surface wave poles dominate at large ρ

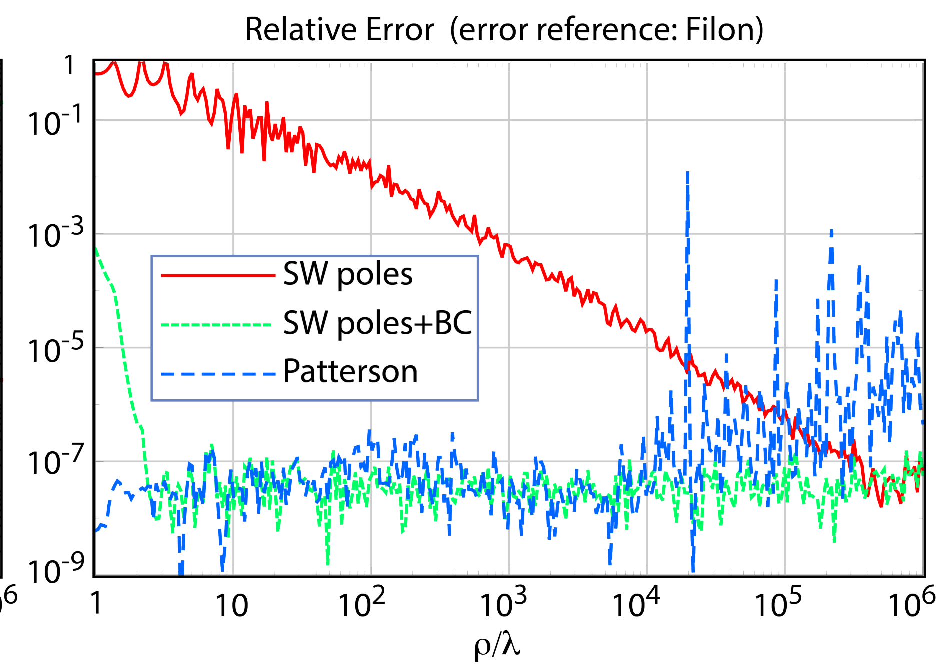


Branch pts.: $k_r = 2\pi,\ 8.886$

SW poles: $k_r = 8.901,\ 13.338,\ 16.736$







Conclusions

- The Filon/Patterson integration needs much fewer evaluations than Patterson only for large ρ

- The number of function evaluations could be reduced further by:

  - Use a global rather than local relative error test
  - Extract surface wave poles
  - Use an scheme such as interpolation to reduce the number of integral evaluations